This is a closed book exam: the only resources allowed are blank paper, pens, and your head. Explain your reasoning. Write clearly, in the sense of logic, language and legibility. The clarity of your explanations affects your grade. All problems have the same weight. Write your name and ID on every solution sheet. Good luck!

1) Plane sweep algorithm in computational geometry

a) Describe briefly the plane sweep algorithm that finds a closest pair among n points in the plane, emphasizing its invariant, and how it achieves its O(n log n) time complexity.

b) Adapt the algorithm above to find a closest pair among n non-overlapping aligned (axis-parallel) rectangles, where the distance between rectangles is measured using the Manhattan metric: v+h as shown at right.
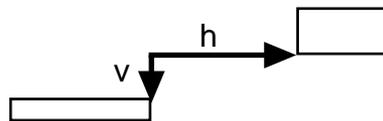

2) Transitive closure

a) Write down the adjacency matrix A of the directed graph shown:

b) Define the operation of transitive closure.
c) Define Boolean matrix multiplication and show how it can be used to compute the connectivity matrix of the digraph in a)
d) Describe Warshall's algorithm for computing the transitive closure, and   e) apply it to the digraph in a)

3) Planar and plane graphs

a) Define the concept "planar graph". State Kuratowski's theorem that characterizes planar graphs. Apply this theorem to Petersen's graph shown here.

b) Illustrate Euler's formula for plane graphs: "Vertices - Edges + Faces - Components = 1" using the example of a disconnected graph consisting of a triangle, an isolated line segment, and an isolated point.

c) Prove Euler's formula.
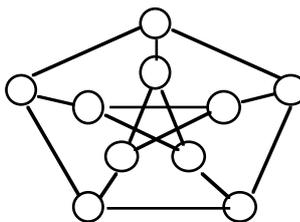
4) Graphs and spanning trees

a) State Cayley's theorem about the number of labeled spanning trees of the complete graph Kn of n labeled vertices.

b) Decode the Pruefer code at right and draw the corresponding labeled tree of 7 vertices

steps	1	2	3	4	5	6
hinge	4	1	1	4	4	7
leaf						

5) Numerics

a) Write down the general solution of the recursion formula $x_{n+1} = -2 x_n - 3/4 x_{n-1}$
b) Write down the special solution that satisfies the initial conditions $x_0 = 1, \ x_1 = -1/2$
c) Compute a few terms of this sequence using formulas a) and b), in a floating point number system that has a single decimal digit in the mantissa, and exponents as large as needed.

6) Traveling salesman problem: Minimum spanning tree tour (MST-tour)

a) Given a complete weighted graph G = (V, E, w: E -> Reals) that satisfies the triangle inequality. Describe the MST-tour approximation to the TSP problem, and prove that the resulting tour is at most twice as long as an optimal tour.
Now assume G is Euclidean, with edge weights given by the Euclidean distance.
b) Prove or disprove: An optimal tour has no crossing edges
c) Prove or disprove: There exists an MST-tour without crossing edges